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**LETTER TO THE EDITOR**

# Self-dual $D$-dimensional quantum Potts model with multi-spin interactions

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**Abstract.** We introduce a family of $D$-dimensional $q$-state quantum Potts models with multi-spin interactions which are self-dual for any $D$. These models are introduced using their $(d=D+1)$-dimensional classical formulations and the transfer matrix technique. The self-duality is proved on the quantum Hamiltonian. In the limit $q\to\infty$ the ground state energy is obtained exactly and the phase transition is first-order. A $1/q$ expansion allows us to get an approximate expression for the line in the $(q, D)$-plane separating regions where the transition changes from first- to second-order.

Let us first consider a generalisation of the classical $q$-state Potts (1952) model for which the Potts variables $n=0, 1, 2, \ldots, q-1$ lie on the $N^d$ vertices of a $d$-dimensional hypercubical lattice. The coupling $K_\tau$ is between first neighbours and of the usual type in the temporal direction, whereas in the $(D=d-1)$-dimensional hypercubical slices there is a coupling $K_\rho$ between the $2^D$ Potts variables lying on the vertices of the $N^D$ $D$-dimensional simplices. The multi-spin interaction has the form introduced by Enting (1975).

The Hamiltonian reads

$$-\beta\mathscr{H}=K_\tau\sum_{\{l(j,m)\}}\{\delta_q[\eta_{l(j,m)}]-1\}+K_\rho\sum_{\{s(j,m)\}}\{\delta_q[\eta_{s(j,m)}]-1/q\} \tag{1}$$

where the first sum runs over the links $l(j, m)$ in the temporal direction and the second sum over the $D$-dimensional simplices $s(j, m)$. $\delta_q(r)$ is a Krönecker delta-function modulo $q$:

$$\delta_q(r)=\frac{1}{q}\sum_{p=0}^{q-1}\cos\left(\frac{2\pi p}{q}r\right). \tag{2}$$

The $\eta$ variables are defined as (see figure 1)

$$\eta_{l(j,m)}=n_{j,m+1}-n_{j,m} \tag{3}$$

for the temporal link $l(j, m)$ joining spin $j$ in the $m$th temporal slice to spin $j$ in the $(m+1)$th temporal slice, and

$$\eta_{s(j,m)}=\sum_{k=0}^{2^D-1}n_{jk,m} \tag{4}$$

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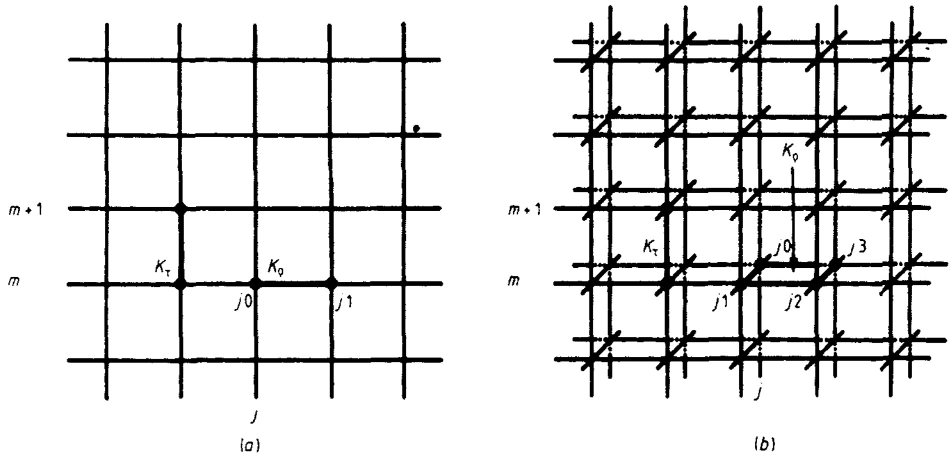


**Figure 1.** The coupling $K_\tau$ is between nearest-neighbour Potts variables in the temporal direction: $K_\rho$ in the temporal slice $m$ couples ($a$) two nearest-neighbour Potts variables $n_{j0}$ and $n_{j1}$ when $d=2$ and ($b$) four variables $n_{j0}$, $n_{j1}$, $n_{j2}$ and $n_{j3}$ around a plaquette when $d=3$.

for the simplex $j$ in the $m$th temporal slice, the sum running over the $2^D$ vertices $jk$ of the simplex.

The multi-spin interaction introduced above reduces to the Ising multi-spin interaction when $q=2$ with the following correspondence:

$$n=0 \qquad \sigma=1; \qquad n=1 \qquad \sigma=-1.$$

We make use of the transfer matrix technique (Lajzerowicz and Pfeuty 1971, Fradkin and Susskind 1978, Kogut 1979) to get the $D$-dimensional quantum Hamiltonian formulation of the model in the $\tau$-continuum limit. When the temporal lattice spacing $\tau$ goes to zero, the transfer matrix becomes

$$\hat{T}=\exp(-\tau\hat{H}) \underset{\tau\to 0}{\sim} \hat{1}-\tau\hat{H}+O(\tau^2). \qquad (5)$$

The transfer matrix connecting to successive slices $m$ and $m+1$ in the temporal direction

$$T_{m,m+1}=\exp[-\mathscr{L}(m, m+1)] \qquad (6)$$

involves the Lagrangian $\mathscr{L}(m, m+1)$ which, according to equations (1), (3) and (4), may be written

$$\mathscr{L}(m, m+1)=-\frac{K_\rho}{2}\sum_j\left[\delta_q\left(\sum_{k=0}^{2D-1} n'_{jk}\right)-\frac{1}{q}\right]-K_\tau\sum_j[\delta_q(n'_j-n_j)-1]$$
$$-\frac{K_\rho}{2}\sum_j\left[\delta_q\left(\sum_{k=0}^{2D-1} n_{jk}\right)-\frac{1}{q}\right] \qquad (7)$$

where $\Sigma_j$ is a sum over the $N^D$ vertices in the temporal slices. Primed variables refer to the $(m+1)$th slice. $\hat{T}$ is a $q^{ND} \times q^{ND}$ matrix with the matrix elements (equations (6), (7))

$$T_{m,m+1} = \exp\left\{-\frac{K_\rho}{2}\sum_j \left[\frac{1}{q} - \delta_q\left(\sum_{k=0}^{2D-1} n'_{jk}\right)\right] - K_\tau \sum_j [1 - \delta_q(n'_j - n_j)]\right.$$
$$\left. - \frac{K_\rho}{2}\sum_j \left[\frac{1}{q} - \delta_q\left(\sum_{k=0}^{2D-1} n_{jk}\right)\right]\right\}. \tag{8}$$

On each vertex $j$ we introduce the cyclic states $|n_j\rangle$ ($n_j = 0, 1, 2, \ldots, q-1$) and unitary operators $\hat{C}_j$ and $\hat{R}_j$ such that

$$\hat{C}_j|n_j\rangle = \exp(\mathrm{i}\,2\pi n_j/q)|n_j\rangle$$
$$\hat{R}_j|n_j\rangle = |n_j+1\rangle \qquad \hat{R}_j^\dagger|n_j\rangle = |n_j-1\rangle$$
$$|n_j+q\rangle \equiv |n_j\rangle. \tag{9}$$

These operators give back the Pauli spin operators $\hat{\sigma}_z$ and $\hat{\sigma}_x$ when $q=2$. On the same site, using equation (9), we get the commutation rules

$$\hat{R}_j\hat{C}_j = \exp(-\mathrm{i}2\pi/q)\hat{C}_j\hat{R}_j$$
$$\hat{R}_j^\dagger\hat{C}_j = \exp(\mathrm{i}2\pi/q)\hat{C}_j\hat{R}_j^\dagger$$
$$\hat{C}^q \equiv \hat{R}^q = \hat{1} \tag{10}$$

whereas the operators commute on different sites. The $T$-matrix may be written as an operator product $\hat{T} = \hat{T}_1\hat{T}_2\hat{T}_1$ involving the operator $\hat{T}_1$ which is diagonal in the basis introduced above and reads

$$\hat{T}_1 = \exp\left\{\frac{K_\rho}{2}\sum_j \frac{1}{2q}\sum_{p=1}^{q-1}\left[\left(\prod_{k=0}^{2D-1}\hat{C}_{jk}\right)^p + \mathrm{HC}\right]\right\} \tag{11}$$

and the operator $\hat{T}_2$ such that

$$\hat{T}_2|_{0\,\text{flip}} = 1 \qquad \hat{T}_2|_{1\,\text{flip}} = \exp(-K_\tau) \qquad \hat{T}_2|_{n\,\text{flips}} = \exp(-nK_\tau). \tag{12}$$

In order to get for $\hat{T}$ the form given in equation (5) in the $\tau$-continuum limit, we are led to take $K_\rho \sim \tau$ and $\tau \sim \exp(-K_\tau)$, i.e. the extreme anisotropic limit $K_\rho \to 0$ and $K_\tau \to \infty$ for the couplings. Then $\hat{T}_2|_{n\,\text{flips}}$ is of order $\tau^n$ and $n$-flips contributions may be ignored when $n \geq 2$. Then

$$\hat{T}_2 = \hat{1} + \frac{\tau}{2q}\sum_j \sum_{p=1}^{q-1}(\hat{R}_j^p + \mathrm{HC}) + \mathrm{O}(\tau^2) \tag{13}$$

where $\tau = q\exp(-K_\tau)$. With $K_\rho = \lambda\tau$ we have

$$\hat{T} = \hat{1} + \frac{\lambda\tau}{2q}\sum_j\sum_{p=1}^{q-1}\left[\left(\prod_{k=0}^{2D-1}\hat{C}_{jk}\right)^p + \mathrm{HC}\right] + \frac{\tau}{2q}\sum_j\sum_{p=1}^{q-1}(\hat{R}_j^p + \mathrm{HC}) + \mathrm{O}(\tau^2) \tag{14}$$

and the $D$-dimensional quantum Hamiltonian reads

$$\hat{H}(\lambda) = -\frac{\lambda}{2q}\sum_j\sum_{p=1}^{q-1}\left[\left(\prod_{k=0}^{2D-1}\hat{C}_{jk}\right)^p + \mathrm{HC}\right] - \frac{1}{2q}\sum_j\sum_{p=1}^{q-1}(\hat{R}_j^p + \mathrm{HC}). \tag{15}$$

The dual lattice may be constructed through a positive shift of the original lattice by half a lattice spacing in each of the $D$ spatial directions. The dual vertices lie in the centre of the simplices of the original lattice (figure 2). Let us keep the same index

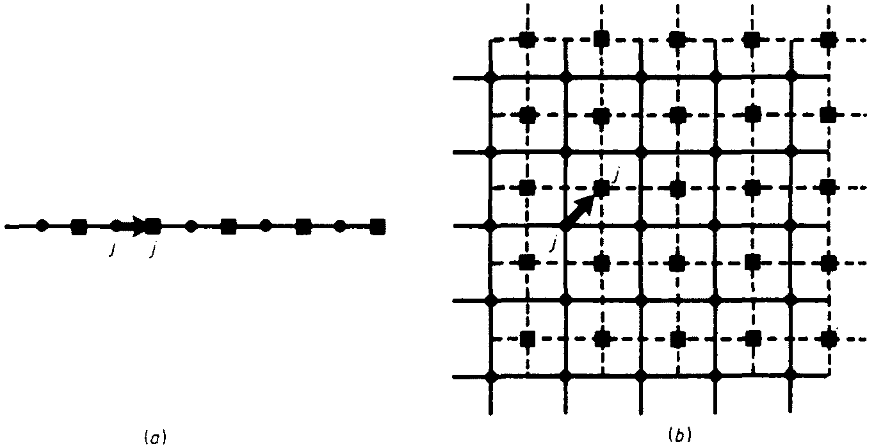


**Figure 2.** Dual lattice: the dual vertices (squares) lie in the centre of the simplices of the original lattice (heavy lines), i.e. (*a*) of the links when $D=1$ or (*b*) of the plaquettes when $D=2$. The arrows give the translations in which original and dual vertices correspond.

$j$ for a vertex of the original lattice and the corresponding one on the dual lattice and let $\hat{\nu}_l$ $(l=1,2,\ldots,D)$ be the basis vectors of the lattices. We define the dual operators as

$$\hat{S}_j^\dagger = \prod_{k=0}^{2D-1} \hat{C}_{jk} \qquad \hat{D}_j = \prod_{n_l,m_l} \hat{R}_{j-\Sigma_{l=1}^D n_l\hat{\nu}_l} \hat{R}^\dagger_{j-\Sigma_{l=1}^D m_l\hat{\nu}_l} \tag{16}$$

with

$$\sum_{l=1}^D n_l \text{ even} \geqslant 0 \qquad \sum_{l=1}^D m_l \text{ odd} > 0$$

and the vertex $j-\Sigma_{l=1}^D n_l\hat{\nu}_l$ is deduced from the vertex $j$ through a translation by a vector $\boldsymbol{t}=-\Sigma_{l=1}^D n_l\hat{\nu}_l$ (see figures 3(*a*), (*b*)). These are unitary operators

$$\hat{S}_j\hat{S}_j^\dagger \equiv \hat{D}_j\hat{D}_j^\dagger = \hat{1}$$

with the same algebra as $\hat{R}$ and $\hat{C}$ (figures 3(*c*), (*d*))

$$\hat{S}_j^\dagger\hat{D}_j = \exp(\mathrm{i}2\pi/q)\hat{D}_j\hat{S}_j^\dagger \tag{17a}$$

$$\hat{S}_j\hat{D}_j = \exp(-\mathrm{i}2\pi/q)\hat{D}_j\hat{S}_j \tag{17b}$$

$$\hat{S}_j^q \equiv \hat{D}_j^q = \hat{1} \tag{17c}$$

and they commute on different sites. Furthermore (figure 3(*e*))

$$\hat{R}_{j+\Sigma_{l=1}^D \hat{\nu}_l} = \prod_{k=0}^{2D-1} \hat{D}_{jk} \tag{18}$$

so that the original Hamiltonian may be rewritten with the dual operators as

$$\hat{H}(\lambda) = -\frac{1}{2q}\sum_j\sum_{p=1}^{q-1}\left[\left(\prod_{k=0}^{2D-1}\hat{D}_{jk}\right)^p + \mathrm{HC}\right] - \frac{\lambda}{2q}\sum_j\sum_{p=1}^{q-1}(\hat{S}_j^p + \mathrm{HC})$$
$$= \lambda\hat{H}(\lambda^{-1}) \tag{19}$$

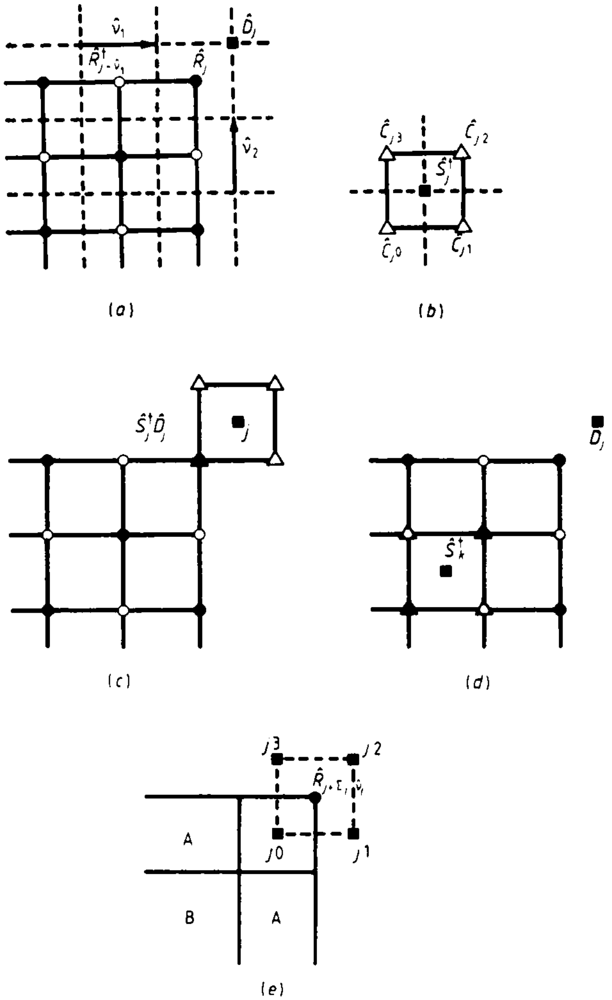


**Figure 3.** Dual operators (here for $D = 2$). (*a*) The dual operator $D_j$ (square) is an infinite product of $\hat{R}$ (full circles) and $\hat{R}^{\dagger}$ (open circles) operators. (*b*) $\hat{S}_j^{\dagger}$ (square) is a product of four $\hat{C}$ operators on the vertices of simplex $j$ (triangles). (*c*) In the operator product $\hat{S}_j^{\dagger}\hat{D}_j$ all operators commute except on site $j$ where $\hat{C}_j\hat{R}_j$ gives the factor $\exp(\mathrm{i}2\pi/q)$. (*d*) In $\hat{S}_k^{\dagger}\hat{D}_j$ $(k \neq j)$ either all operators commute or the operator products $\hat{C}\hat{R}^{\dagger}$ and $\hat{C}\hat{R}$ enter in pairs leading to a cancellation of the exponential factors. (*e*) In the operator product $\Pi_{k=0}^{2^D-1}\hat{D}_{jk}$ (squares), $\hat{R}_{j+\Sigma_{l=1}^{D}\hat{v}_l}$ (full circle) enters only once whereas all other $\hat{R}$ and $\hat{R}^{\dagger}$ operators enter the product in pairs under the form $(\hat{R}\hat{R}^{\dagger})^p = 1$ or $(\hat{R}^{\dagger}\hat{R})^p = 1$. Here $p = 1$ in region A and $p = 2$ in region B.

since $\hat{H}$ takes the same form with $\hat{D}$ and $\hat{S}$ as with $\hat{C}$ and $\hat{R}$. It follows that $\hat{H}$ is self-dual for all $q$ and $D$, and when there is a unique phase transition it occurs at the critical coupling $\lambda_c = 1$.

According to equation (19), the ground state energy per site when $\lambda < 1$, $\varepsilon_<(\lambda)$, is related to its value in the low-temperature phase ($\lambda > 1$) through

$$\varepsilon_<(\lambda) = \lambda\varepsilon_>(\lambda^{-1}). \qquad (20)$$

Working in the basis where the $\hat{C}$ operators are diagonal, appropriate to the low-temperature phase, the Hamiltonian may be rewritten as

$$\hat{H} = N^D\lambda/q + \hat{H}_0 + \hat{V} \tag{21a}$$

$$\hat{H}_0 = -\lambda \sum_j \delta_q\left(\sum_{k=0}^{2D-1} n_{jk}\right) \tag{21b}$$

$$\hat{V} = -\frac{1}{q}\sum_j \sum_{p=1}^{q-1} \hat{R}_j^p. \tag{21c}$$

In this basis $\hat{R}_j$ flips the Potts state on site $j$. The $g_D$ degenerate ground states of $\hat{H}_0$ are not coupled by $\hat{V}$ to any finite order in perturbation theory, and we may proceed to a perturbation expansion on the ground state $|0\rangle$ where all the Potts variables are in the same state $n_j = 0$ for all $j$.

Going up to terms of order $1/q$ (Kogut and Sinclair 1981) we get

$$\varepsilon_>(\lambda) = -\lambda + \frac{1}{q}\left(\lambda - \frac{1}{2^D\lambda - 1}\right) + \mathrm{O}\left(\frac{1}{q^2}\right) \tag{22a}$$

and using equation (20)

$$\varepsilon_<(\lambda) = -1 + \frac{1}{q}\left(1 - \frac{\lambda^2}{2^D - \lambda}\right) + \mathrm{O}\left(\frac{1}{q^2}\right). \tag{22b}$$

In the limit $q \to \infty$ the transition is first-order for all $D$ with a latent heat

$$\Delta = \lim_{q\to\infty}\left(\frac{\partial\varepsilon_<}{\partial\lambda} - \frac{\partial\varepsilon_>}{\partial\lambda}\right)_{\lambda=\lambda_c=1} = 1. \tag{23}$$

Up to terms of order $1/q$ we get

$$\Delta = 1 - \frac{1}{q}\frac{2^D(2^D+1)}{(2^D-1)} + \mathrm{O}\left(\frac{1}{q^2}\right) \tag{24}$$

so that to this order the latent heat vanishes on the line

$$q_c(D) = \frac{2^D(2^D+1)}{(2^D-1)^2} \tag{25}$$

in the $(q, D)$-plane (figure 4), which is the frontier between first-order (on the high-$q$ side) and second-order regions. In agreement with the result previously obtained by Kogut *et al* (1980) for the quantum Potts chain, equation (25) gives $q_c(1) = 6$, whereas the exact result is known to be $q_c(1) = 4$ (Baxter 1973). When $D = 0$, the classical counterpart is a $q$-state Potts model on the linear chain with an external field, the quantum Hamiltonian is easily diagonalised and the ground state energy is

$$\varepsilon = -\frac{1}{2q}\{(q-2)(\lambda+1) + [q^2(\lambda-1)^2 + 4q\lambda]^{1/2}\}. \tag{26}$$

Up to terms of first order in $1/q$ this result agrees with equations (22$a$) and (22$b$) above.

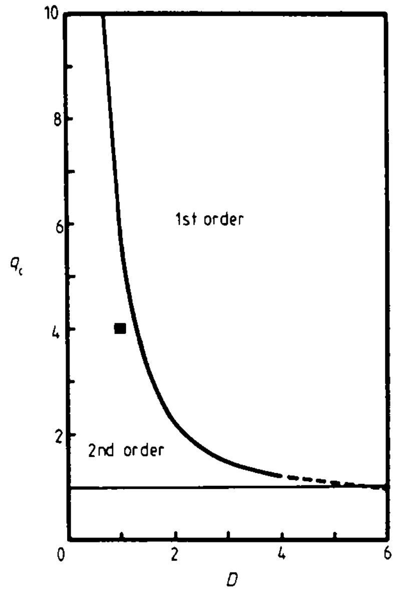


**Figure 4.** Approximate frontier of the first-order region in the $(q, D)$-plane on which the latent heat vanishes up to terms of order $1/q$. The point $(q=4, D=1)$ belongs to and the line $q=1$ (percolation limit) is an asymptote for the exact frontier.

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